Supplemental Notes

EE503 Week 06 Dr. Franzke

HW # 6

① Leon-Garcia 4.39-4.40, 4.53-4.57, 4.70

Topics

① Moments

ⓐ Population statistics

population mean: $\mu_X = E_X[\underset{\text{r.v.}}{X}] = \begin{cases} \sum_x x \cdot f(x) & \text{discrete} \\ \int_{-\infty}^{\infty} x \cdot f(x)\,dx & \text{continuous} \end{cases}$

if sum | integral exists (i.e. finite)

$E_X[X]$ "averages out the randomness"

ⓑ vs. Sample statistics

sample mean: $\underset{\text{r.v.}}{\bar{X}_n} = \frac{1}{n} \sum_{k=1}^{n} \underset{\text{r.v.}}{X_k}$

(realizations: $\bar{x}_n = \frac{1}{n} \sum_{k=1}^{n} x_k$)

Defn: The population mean (expectation)

$$\mu_X = E_X[X] = \begin{cases} \sum_x x \cdot f_X(x) & X \text{ discrete} \\ \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx & X \text{ continuous} \end{cases}$$

constant (μ_X), r.v. (X)

if sum/integral exists (is finite)

Defn: The population variance

$$\sigma_X^2 = V_X[X] = E_X\left[(X - E_X[X])^2\right].$$

if finite ($\sigma_X^2 < \infty$)

Note: Cauchy $\sigma_X^2 = \infty$.

Defn: The k^{th} higher order moment

$$E_X[X^k] = \int_{-\infty}^{\infty} x^k \cdot f_X(x)\,dx \quad (\text{if} < \infty)$$

In general: k^{th} moment exists $\underset{\not\Leftarrow}{\longrightarrow}$ $(k-1)^{th}$ moment exists

$\therefore \quad \sigma_X^2 < \infty \implies |\mu_X| < \infty$

Thm: If $k \le m$ and $k \in \mathbb{Z}^+$ and $m \in \mathbb{Z}^+$ then

$E[X^m]$ exists $\longrightarrow$ $E[X^k]$ exists.

Pf: (continuous case, replace integrals with sums for discrete)

$$\int_{-\infty}^{\infty} |x|^k f_X(x)\,dx = \int_{|x|\le 1} |x|^k \cdot f_X(x)\,dx + \int_{|x|>1} |x|^k \cdot f_X(x)\,dx$$

$$\le \int_{|x|\le 1} |x|^k \cdot f_X(x)\,dx + \int_{|x|>1} |x|^m \cdot f_X(x)\,dx$$

since $k \le m$ and $|x|>1$

$$\le \int_{|x|\le 1} f_X(x)\,dx + \int_{|x|>1} |x|^m f_X(x)\,dx.$$

since $|x|\le 1 \longrightarrow |x|^k$

$$\le \underbrace{\int_{-\infty}^{\infty} f_X(x)\,dx}_{=1\ (\text{pdf})} + \int_{|x|>1} |x|^m \cdot f_X(x)\,dx$$

since $\{|x|\le 1\} \subset \mathbb{R}$

$$\le 1 + \int_{-\infty}^{\infty} |x|^m \cdot f_X(x)\,dx.$$

since $\{|x|>1\} \subset \mathbb{R}$

$$= 1 + E_X[|X|^m].$$

$$< \infty$$ since $E[X^m]$ exists by hypo.

$\therefore$ $E[X^k]$ exists.

QED.

Ex: exponential moments $X \sim \exp(\theta)$

Thrm. $\boxed{E[X^k] = k!\,\theta^k \text{ if } X \sim \exp(\theta)}$

Pf: $E[X^k] = \int_{-\infty}^{\infty} x^k \cdot f_X(x)\,dx.$

$= \int_0^{\infty} x^k \cdot \left(\frac{1}{\theta} \cdot e^{-x/\theta}\right) dx \quad \frac{\theta^k}{\theta^k}$

since $X \sim \exp(\theta)$

$= \theta^{k-1} \int_0^{\infty} \left(\frac{x}{\theta}\right)^k e^{-x/\theta}\, d\theta$

$\therefore$ let $u = \frac{x}{\theta}$ $\quad du = \frac{dx}{\theta}.$

$\therefore dx = \theta \cdot du$ $\quad x = 0 \longrightarrow u = 0$, $x = \infty \longrightarrow u = \infty$

$= \theta^{k-1} \cdot \int_0^{\infty} u^k e^{-u}\, \theta \cdot du$

$= \theta^k \cdot \underbrace{\int_0^{\infty} u^{(k+1)-1} e^{-u}\, du}_{\Gamma(k+1)}$

$= \theta^k \cdot \Gamma(k+1)$

$= \theta^k \cdot k!$ since $\Gamma(\alpha+1) = \alpha \cdot \Gamma(\alpha).$

Corr: $k! = E[X^k]$ if $X \sim \exp(1).$

Proposition 1: If $Y = aX + b$ $(a \neq 0)$ and $X \sim f_X$

then. ① $E_Y[Y] = a \cdot E_X[X] + b. = a \cdot \mu_X + b$

② $V_Y[Y] = a^2 \cdot V_X[X] = a^2 \cdot \sigma_X^2$

Proposition 2: $V_X[X] = E_X[X^2] - E_X^2[X].$

Proposition 3: ("Standardization") If $Z = \frac{X-\mu}{\sigma}$ $(\sigma_X^2 < \infty)$

then: ① $E[Z] = 0$

② $V[Z] = 1$

Ex: If $X \sim \exp(\theta)$, then

$$\mu_X = E[X^1] = \theta$$

$$\therefore \sigma_X^2 = E[X^2] - E^2[X] = 2\theta^2 - \theta^2 = \theta^2$$

More generally:

Thrm: $E[X^k] = \theta^k \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}$ $\begin{matrix} x>0 \\ \alpha>0 \\ \theta>0 \end{matrix}$ if $X \sim \gamma(\alpha, \theta)$

Prf: $$E[X^k] = \int_{-\infty}^{\infty} x^k \cdot f_X(x)\, dx$$

$$= \frac{1}{\Gamma(\alpha)\theta^\alpha} \cdot \int_0^\infty x^k \cdot x^{\alpha-1} e^{-x/\theta}\, dx$$

$$= \frac{1}{\Gamma(\alpha)\,\theta^{\alpha}} \int_0^{\infty} x^{(\alpha+k)-1} e^{-x/\theta}\, dx.$$

let $x = u \cdot \theta$ $\quad dx = \theta \cdot du.$

$$= \frac{1}{\Gamma(\alpha)\cdot\theta^{\alpha}} \int_0^{\infty} (u\cdot\theta)^{(\alpha+k)-1} e^{-u}\, \theta\, du$$

$$= \frac{\theta^{(\cancel{\alpha}+k)}}{\Gamma(\alpha)\cdot\cancel{\theta^{\alpha}}} \cdot \underbrace{\int_0^{\infty} u^{(\alpha+k)-1} e^{-u}\, du.}_{\Gamma(\alpha+k)}$$

$$= \theta^{k} \cdot \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}$$

QED.

Checking $\alpha = 1$: $\Gamma(\alpha+k) = \Gamma(k+1) = k\cdot\Gamma(k) = k!$

$\therefore E[X^k] = \theta^k\, k!$ if $X \sim Exp(\theta)$

Defn: (Gamma function) $\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x}\, dx$ $\quad \alpha > 0.$

$\therefore \Gamma(1) = \int_0^{\infty} e^{-x}\, dx = 1.$

Thrm: $\Gamma(\alpha+1) = \alpha\,\Gamma(\alpha)$ if $\alpha > 0.$

Prf: $\Gamma(\alpha+1) = \int_0^{\infty} e^{-x} x^{(\alpha+1)-1}\, dx.$

$$= \int_0^{\infty} e^{-x} x^{\alpha}\, dx$$

(integration by parts)

let $u = x^{\alpha}$ $\quad dv = e^{-x} dx$

$du = \alpha x^{\alpha-1}$ $\quad v = -e^{-x}$

$$= u \cdot v \Big|_0^\infty - \int_0^\infty v \cdot du$$

$$= \underbrace{\lim_{x \to \infty} \frac{-x^\alpha}{e^x}}_{= 0 \text{ by L'Hospital}} - 0 + \alpha \cdot \int_0^\infty e^{-x} x^{\alpha-1} \, dx.$$

$$= \alpha \cdot \int_0^\infty e^{-x} \, x^{\alpha-1} \, dx$$

$$= \alpha \cdot \Gamma(\alpha).$$

QED.

Corr: $\Gamma(n+1) = n!$ if $n \in \mathbb{Z}^+$

Corr: ① $\mu_x = \alpha\theta$ ② $\sigma_x^2 = \alpha\theta^2$ if $X \sim \gamma(\alpha, \theta)$

Pf: $$\mu_x = E[X^1] \overset{X \sim \gamma}{=} \theta^1 \cdot \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)}$$

$$= \theta \cdot \frac{\alpha \cdot \cancel{\Gamma(\alpha)}}{\cancel{\Gamma(\alpha)}}$$

$$= \alpha \, \theta.$$

$$\sigma_x^2 = E[X^2] - E^2[X] = \theta^2 \cdot \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)} - \alpha^2 \cdot \theta^2.$$

$$= \theta^2 \left[\frac{(\alpha+1)\Gamma(\alpha+1)}{\Gamma(\alpha)} - \alpha^2 \right].$$

$$= \theta^2 \cdot \left[\frac{(\alpha+1)\alpha \cdot \cancel{\Gamma(\alpha)}}{\cancel{\Gamma(\alpha)}} - \alpha^2 \right].$$

$$= \theta^2 \left[\alpha^2 + \alpha - \alpha^2 \right].$$

$$= \alpha \cdot \theta^2.$$

QED

Defn: Beta function $B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx$ $\quad \alpha>0,\ \beta>0$

$$= \frac{\Gamma(\alpha)\cdot\Gamma(\beta)}{\Gamma(\alpha+\beta)} \quad \text{(see below)}$$

Thrm: $E[X^k] = \dfrac{B(\alpha+k,\beta)}{B(\alpha,\beta)}$ if $X \sim \text{Beta}(\alpha,\beta)$ and $k>0$

Pf:

$$E[X^k] = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cdot\Gamma(\beta)} \underbrace{\int_0^1 x^{(\alpha+k)-1}(1-x)^{\beta-1}\,dx}_{B(\alpha+k,\beta)}$$

$$= \frac{B(\alpha+k,\beta)}{B(\alpha,\beta)}$$

Corr: $E[X^k] = \dfrac{\alpha+k-1}{\alpha+\beta+k-1}\cdot E[X^{k-1}]$ $\quad (k\geq 1).$

Pf:

$$E[X^k] \overset{\text{thrm}}{=} \frac{B(\alpha+k,\beta)}{B(\alpha,\beta)}$$

$$= \frac{\Gamma(\alpha+k)\cdot\Gamma(\beta)}{\Gamma(\alpha+\beta+k)}\cdot\frac{1}{B(\alpha,\beta)}$$

$$= \frac{(\alpha+k-1)\cdot\Gamma(\alpha+k-1)\cdot\Gamma(\beta)}{(\alpha+\beta+k-1)\Gamma(\alpha+\beta+k-1)}\cdot\frac{1}{B(\alpha,\beta)}$$

$$= \frac{\alpha+k-1}{\alpha+\beta+k-1}\cdot\frac{B(\alpha+k-1,\beta)}{B(\alpha,\beta)}$$

$$= \frac{\alpha+k-1}{\alpha+\beta+k-1}\cdot E[X^{k-1}]$$

QED.

Corr. If $X \sim \text{Beta}(\alpha, \beta)$

① $\mu_x = \dfrac{\alpha}{\alpha+\beta}.$

② $\sigma_x^2 = \dfrac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$

Pf:

$$\mu_x = E[X^1] = \frac{B(\alpha+1, \beta)}{B(\alpha, \beta)}$$

$$= \frac{\Gamma(\alpha+1)\,\Gamma(\beta)}{\Gamma(\alpha+\beta+1)} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)}$$

$$= \frac{\alpha \cdot \cancel{\Gamma(\alpha)}\,\cancel{\Gamma(\beta)}}{(\alpha+\beta)\cdot\cancel{\Gamma(\alpha+\beta)}} \cdot \frac{\cancel{\Gamma(\alpha+\beta)}}{\cancel{\Gamma(\alpha)}\,\cancel{\Gamma(\beta)}}$$

$$= \frac{\alpha}{\alpha+\beta}.$$

$$E[X^2] = \frac{B(\alpha+2, \beta)}{B(\alpha, \beta)} = \frac{\Gamma(\alpha+2)\cdot\cancel{\Gamma(\beta)}}{\Gamma(\alpha+\beta+2)} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cdot\cancel{\Gamma(\beta)}}$$

$$= \frac{(\alpha+1)\cdot\Gamma(\alpha+1)}{(\alpha+\beta+1)\cdot\Gamma(\alpha+\beta+1)} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}$$

$$= \frac{(\alpha+1)\,\alpha\cdot\cancel{\Gamma(\alpha)}}{(\alpha+\beta+1)(\alpha+\beta)\cdot\cancel{\Gamma(\alpha+\beta)}} \cdot \frac{\cancel{\Gamma(\alpha+\beta)}}{\cancel{\Gamma(\alpha)}}$$

$$= \frac{\alpha(\alpha+1)}{(\alpha+\beta+1)(\alpha+\beta)}.$$

$$\therefore \sigma_x^2 = E[X^2] - E^2[X] = \frac{\alpha(\alpha+1)}{(\alpha+\beta+1)(\alpha+\beta)} - \frac{\alpha^2}{(\alpha+\beta)^2}$$

$$= \frac{\alpha(\alpha+1)(\alpha+\beta) - \alpha^2(\alpha+\beta+1)}{(\alpha+\beta+1)(\alpha+\beta)^2} = \frac{\cancel{\alpha^3} + \cancel{\alpha^2\beta} + \cancel{\alpha^2} + \alpha\beta - \cancel{\alpha^3} - \cancel{\alpha^2\beta} - \cancel{\alpha^2}}{(\alpha+\beta+1)(\alpha+\beta)^2}$$

$$= \frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$$

QED

Normal moments

☆ **Thrm:** If $Z \sim N(0,1)$ (standard normal)

then
$$E[Z^k] = \begin{cases} 0 & \text{if } k \text{ odd} \\ (k-1)(k-3)\cdots 5\cdot 3\cdot 1 & \text{if } k \text{ even} \end{cases}$$

In general: If $X \sim N(\mu, \sigma^2)$ then

$$\underbrace{E[(X-\mu_x)^k]}_{k^{\text{th}} \text{ "central moment"}} = \begin{cases} 0 & \text{if } k \text{ odd} \\ \sigma^2 (k-1)(k-3)\cdots 5\cdot 3\cdot 1 & \text{if } k \text{ even} \end{cases}$$

Prf: Case 1, k even

$$E[Z^k] = \frac{1}{\sqrt{2\pi}} \int_{z=-\infty}^{z=\infty} z^k e^{-z^2/2}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^{k-1} \left(z \cdot e^{-z^2/2}\, dz \right)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u \cdot dv$$ if $u = z^{k-1}$, $dv = ze^{-z^2/2} dz$, $du = (k-1)z^{k-2} dz$, $v = e^{-z^2/2}$

integration by parts

$$= \frac{1}{\sqrt{2\pi}} \left(uv \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} v \cdot du \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(\underbrace{\lim_{z\to\infty} z^{k-1} \cdot e^{-z^2/2} \Big|_{-z}^{z}}_{=0,\ L'Hospital's} + (k-1) \cdot \int_{-\infty}^{\infty} z^{k-2} e^{-z^2/2} dz \right)$$

$$= \frac{1}{\sqrt{2\pi}} (k-1) \cdot \int_{-\infty}^{\infty} z^{k-2} e^{-z^2/2} \, dz$$

$$= (k-1) \cdot E[X^{k-2}]$$

k-even

$$= (k-1)(k-3)(k-5) \cdots 5 \cdot 3 \cdot 1$$

Case 2; k odd

$e^{-z^2/2}$ even function

$\therefore z^k e^{-z^2/2}$ odd function if k odd

$$\therefore \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^k e^{-z^2/2} \, dz = 0.$$

Thrm: If $X \sim G(p)$ $\quad \mu_X = 1/p$, $\sigma_X^2 = q/p^2$ $\quad$ Geometric $0 < p < 1$

$\therefore$ If $X \sim NB$: $\quad \mu_X = k/p$, $\sigma_X^2 = kq/p^2$ $\quad$ Negative binomial

since $X = X_1 + \cdots + X_k$ and independent $X_j \sim G(p)$.

Prf: ① $\mu_X = E[X] \overset{G(p)}{=} \sum_{n=1}^{\infty} n \cdot p \cdot q^{n-1} = p \sum_{n=1}^{\infty} n \cdot q^{n-1}$

$$= p \cdot \sum_{n=1}^{\infty} \frac{d}{dq} q^n \overset{R.O.C.}{=} p \cdot \frac{d}{dq} \left(\sum_{n=1}^{\infty} q^n \right)$$

since R.O.C. $p < 1$

$\therefore$ uniform convergence ($\therefore$ commute)

$$= p\cdot \frac{d}{dq}\left(\frac{q}{1-q}\right) = p\cdot \frac{(1-\cancel{q})-\cancel{q}}{(1-q)^2}$$

$$= \frac{p}{(1-q)^2} \overset{p=1-q}{=} \frac{p}{p^2} = \frac{1}{p} \quad \text{since } p>0$$

② $$E[X^2] = \sum_{n=1}^{\infty} n^2 \cdot p \cdot q^{n-1} = p\cdot \sum_{n=1}^{\infty} n^2 q^{n-1}$$

$$= p\cdot \sum_{n=1}^{\infty} \frac{d}{dq}(n\cdot q^n) \overset{\substack{R.O.C.\\ u.c.}}{=} p\cdot \frac{d}{dq}\left(\sum_{n=1}^{\infty} n\cdot q^n\right)$$

$$= p\cdot \frac{d}{dq}\left(q\cdot \sum_{n=1}^{\infty} n\cdot q^{n-1}\right)$$

$$= p\cdot \frac{d}{dq}\left(\frac{q}{p} \sum_{n=1}^{\infty} n\cdot p\cdot q^{n-1}\right)$$

$$= p\cdot \frac{d}{dq}\cdot\left(\frac{q}{p}\cdot E[X]\right)$$

$$\overset{①}{=} p\cdot \frac{d}{dq}\left(\frac{q}{p^2}\right) = p\cdot \frac{d}{dq}\left(\frac{q}{(1-q)^2}\right)$$

$$= p\cdot \frac{(1-q)^2 + 2q(1-q)}{(1-q)^4} \quad \text{Quotient rule}$$

$$= \frac{p}{(1-q)^2} + \frac{2pq}{(1-q)^3} = \frac{p}{p^2} + \frac{2p\cancel{q}}{p^3}$$

$$\overset{p>0}{=} \frac{1}{p} + \frac{2(1-p)}{p^2} = \frac{1}{p} + \frac{2}{p^2} - \frac{2p}{p^2} = \frac{1}{p} + \frac{2}{p^2} - \frac{2}{p}$$

$$= \frac{2}{p^2} - \frac{1}{p}$$

$$\therefore \sigma_X^2 = E[X^2] - E^2[X] = \frac{2}{p^2} - \frac{1}{p} - \frac{1}{p^2}$$

$$= \frac{1}{p^2} - \frac{1}{p} = \frac{1-p}{p^2}$$

$$= \frac{q}{p^2}$$

QED.

Thrm: If $X \sim b(n,k,p)$ ① $E[X] = n \cdot p$ **Binomial**
② $V[X] = n \cdot p \cdot q$

Prf: ① $E[X] = \sum_{k=0}^{n} k \cdot P[X=k] = \sum_{k=0}^{n} k \cdot b(n,k,p)$

$$\overset{X\sim b}{=} \sum_{k=0}^{n} k \binom{n}{k} p^k q^{n-k} = \sum_{k=1}^{n} k \cdot \frac{n!}{k!(n-k)!} p^k q^{n-k}$$

$$= \sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!} p^k q^{n-k}$$ put $j = k-1$ $\therefore k = j+1$

$$= \sum_{j+1=1}^{j+1=n} \frac{n(n-1)!}{j!(n-1-j)!} p^{j+1} q^{(n-1)-j}$$

$$= \sum_{j=0}^{n-1} \frac{n \cdot (n-1)!}{j!((n-1)-j)!} p \cdot p^j q^{(n-1)-j}$$

$$= n \cdot p \underbrace{\sum_{j=0}^{n-1} \frac{(n-1)!}{j!((n-1)-j)!} p^j q^{(n-1)-j}}_{\text{binomial theorem } = 1}$$

$$= n \cdot p.$$

② $E[X^2] = \sum_{k=0}^{n} k^2 \cdot \frac{n!}{k!(n-k)!} p^k q^{n-k}.$

$$= n \cdot p \cdot \sum_{k=1}^{n} k \cdot \frac{(n-1)!}{(k-1)!(n-k)!} p^{k-1} q^{n-k}$$ put $j = k-1$, $k = j+1$

$$= n \cdot p \cdot \sum_{j=0}^{n-1} (j+1) \frac{(n-1)!}{j!((n-1)-j)!} p^j q^{(n-1)-j}$$

$$= np \cdot \Bigg[\underbrace{\sum_{j=0}^{n-1} j \cdot \frac{(n-1)!}{j!((n-1)-j)!} p^j q^{(n-1)-j}}_{= (n-1)p \text{ by } ①} + \underbrace{\sum_{j=0}^{n-1} \frac{(n-1)!}{j!((n-1)-j)!} p^j q^{(n-1)-j}}_{= 1 \text{ by BT.}} \Bigg].$$

$$= np \cdot [(n-1)p + 1] = np(np + (1-p))$$

$$= np(np+q) = (np)^2 + npq$$

$$\therefore V[X] = E[X^2] - E^2[X]$$

$$= \cancel{(np)^2} + npq - \cancel{(np)^2}$$

$$= npq$$

QED.

Thrm: ① $E[I_A] = \overset{p}{P(A)}$

② $V[I_A] = \overset{p}{P(A)} \cdot \overset{q}{P(A^c)}$

for any $A \in \mathcal{Q}$ in probability space $(\Omega, \mathcal{Q}, P)$

Prf: ① $E[I_A] = \int_\Omega I_A(\omega)\, dP$

since $\Omega = A \cup A^c$

$$= \int_A \underbrace{I_A(\omega)}_{=1\ \omega \in A} dP + \int_{A^c} \underbrace{I_{A^c}(\omega)}_{=0\ \omega \notin A} dP$$

$$= \int_A 1 \cdot dP + \underbrace{\cancel{\int_{A^c} 0 \cdot dP}}_{=0} = \int_A dP$$

$$= P(A)$$

② $V[I_A] = \int_\Omega (I_A(\omega) - E[I_A])^2\, dP.$

$$= \int_\Omega (I_A(\omega) - P(A))^2\, dP$$

$$= \int_\Omega (I_A^2(\omega) + P^2(A) - 2 \cdot P(A) \cdot I_A(\omega))\, dP$$

$$= \int_\Omega I_A^2(\omega)\, dP + P^2(A) \int_\Omega dP - 2 \cdot P(A) \cdot \int_\Omega I_A(\omega)\, dP$$

$$= \int_\Omega \underbrace{I_A(\omega)\,dP}_{\text{since } I_A^2 = I_A \text{ for binary sets (not fuzzy)}} + P^2(A) - 2P^2(A)$$

$$= P[A] - P^2[A] = P(A)\cdot(1 - P(A))$$

$$= P(A)\cdot P(A^c)$$

QED

★ **Thrm:** $B(\alpha, \beta) = \int_0^1 u^{\alpha-1}(1-u)^{\beta-1}\,du$ $\quad \alpha>0, \beta>0$

$$= \frac{\Gamma(\alpha)\cdot\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

Prf: $\Gamma(\alpha)\cdot\Gamma(\beta) = \int_{x=0}^{x=\infty} e^{-x}x^{\alpha-1}\,dx \int_{y=0}^{y=\infty} e^{-y}y^{\beta-1}\,dy$

$$\overset{\text{Fubini}}{=} \int_{y=0}^{y=\infty}\left[\int_{x=0}^{x=\infty} e^{-(x+y)}x^{\alpha-1}y^{\beta-1}\,dx\right]dy$$

$$= \int_{y=0}^{y=\infty}\int_{x=0}^{x=\infty} f(x(u,v), y(u,v))\,dx\,dy$$

Let: $x = x(u,v) = uv \qquad y = y(u,v) = u(1-v)$ $\quad$ double substitution

Given: $0 < x < \infty$ and $0 < y < \infty$

$\therefore\ 0 < x+y = u\cdot v + u(1-v) = \cancel{uv} + u - \cancel{uv} = u$

$\therefore\ u > 0$

$\therefore\ x > 0 \longrightarrow uv > 0$

$\therefore\ v > 0$ since $u > 0$

$\therefore\ u < \infty$ since $x = u\cdot v < \infty$ and $v > 0$

$\therefore\ 0 < u < \infty \longleftarrow$ first limit of integration

$\therefore\ y > 0 \longrightarrow u(1-v) > 0 \qquad \therefore\ 1-v > 0$ (since $u > 0$)

$\therefore\ v < 1 \qquad \therefore\ 0 < v < 1 \longleftarrow$ second limit of integration

$$\therefore \Gamma(\alpha)\cdot\Gamma(\beta) \overset{\text{CVT}}{=} \int_{v=0}^{v=1}\int_{u=0}^{u=\infty} f(u,v)\cdot\left|\frac{\partial(x,y)}{\partial(u,v)}\right| \, du\, dv$$

(change of variable theorem)

$$= \int_{v=0}^{v=1}\int_{u=0}^{u=\infty} f(u,v)\ u \cdot du\, dv.$$

since $\underbrace{\left|\frac{\partial(x,y)}{\partial(u,v)}\right|}_{\text{absolute determinant of Jacobian matrix}} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix}$ since $x = u \cdot v.$ $y = u - uv$

$$= \left| -\cancel{vu} - u + \cancel{vu} \right| = |-u| = u.$$

$$= \int_{v=0}^{v=1}\int_{u=0}^{u=\infty} e^{-u} (uv)^{\alpha-1} \left(u(1-v)\right)^{\beta-1} u \ du\, dv$$

$$= \int_{v=0}^{v=1}\int_{u=0}^{u=\infty} e^{-u} u^{\alpha-1} v^{\alpha-1} u^{\beta-1} (1-v)^{\beta-1} u^{1} \, du\, dv$$

$$= \left[\int_{u=0}^{u=\infty} e^{-u} u^{\alpha-1+\beta-1+1} du\right]\cdot\left[\int_{v=0}^{v=1} v^{\alpha-1}(1-v)^{\beta-1} dv\right]$$

$$= \left[\int_{u=0}^{u=\infty} e^{-u} u^{(\alpha+\beta)-1} du\right]\cdot\left[\int_{v=0}^{v=1} v^{\alpha-1}(1-v)^{\beta-1} dv\right]$$

$$= \Gamma(\alpha+\beta)\cdot B(\alpha,\beta)$$

$$\therefore B(\alpha,\beta) = \frac{\Gamma(\alpha)\cdot\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

QED